6.4

These Series Just Go On . . . And On . . . And On . . .

Infinite Geometric Series

LEARNING GOALS

In this lesson, you will:

- Write a formula for an infinite geometric series.
- Compute an infinite geometric series.
- Draw diagrams to model infinite geometric series.
- Determine whether series are convergent or divergent.
- Use a formula to compute a convergent infinite geometric series.

KEY TERMS

- convergent series
- divergent series

Infinity is a concept that philosophers and mathematicians have struggled with for centuries. Infinity is a very abstract idea. How can something be limitless? What does it mean for something to go on forever?

The following quote is from an Indian philosophical text dating back to the 4^{th} or 3^{rd} century B.C.

If you remove a part from infinity or add a part to infinity, still what remains is infinity.

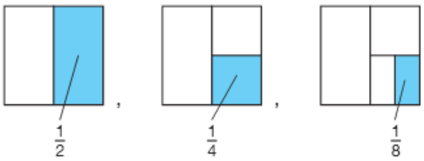
How can this be so?

What does infinity mean to you?

PROBLEM 1 Hang On Bessie, We're Almost There

Previously, you calculated sums of finite series. What if a series was infinite? Let's see if there is a way to calculate the sum of an infinite series.

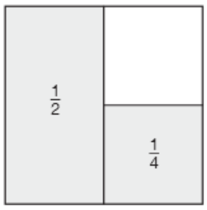
The first three terms of an infinite sequence are represented by the figures shown. In this sequence, each square represents a unit square, and the shaded part represents area.



1. Sketch the next two figures to model this sequence, and write the numbers that correspond to each term.

2. Is this sequence arithmetic, geometric, or neither? Explain how you know. If possible, write an explicit formula for the sequence.

3. Consider the series, or sum, of the first two terms of this infinite sequence. The sum of the first two terms can be modeled with a diagram, as shown.



Continue shading the diagram to represent the sum of the first five terms of the series. What happens to the total area that is shaded every time you shade another piece of the unit square?

4. In the table shown, n represents the term number of the series, and S_n represents the sum of the first n terms of the series. Use the sequence from the unit square in Question 3 to answer each question.

n	1	2	3	4	5	10	25
S_n as a Fraction							
S_n as a Decimal							

a. Complete the table for $n = 1$ through $n = 5$ to show the sum of the series that corresponds to the previous diagram. Write each sum as a fraction and as a decimal.

b. Describe the pattern you see in the table.

6

c. Use the pattern to complete the table for the final two columns.

5. What value does the series approach as n gets greater?

In the figure that models this series, each additional part of the unit square is one half of the previous part. If you could continue to add these parts forever, the unit square would eventually be filled.

Likewise, in the table of values that models this series, the sums get closer and closer to 1 as n gets greater. Therefore, you can say that this infinite geometric series is equal to 1.

6. Miley and Damian determined formulas they could use to compute the first n-terms of the series.

Miley

I noticed that when each sum is written as a fraction, the denominator is equal to 2^n and the numerator is one less than the denominator.

So, I can calculate the first n-terms of the series by using the formula shown.

$$S_n = \frac{2^n - 1}{2^n}$$

Damian

I know that $S_n = \frac{g_n \cdot r - g_1}{r - 1}$ can be used to compute the first n-terms of any geometric series.

For this series, $g_n = \left(\frac{1}{2}\right)^n$, $g_1 = \frac{1}{2}$, and $r = \frac{1}{2}$.

Substituting these values gives:

$$S_n = \frac{\left(\frac{1}{2}\right)^n \cdot \frac{1}{2} - \frac{1}{2}}{\frac{1}{2} - 1}$$

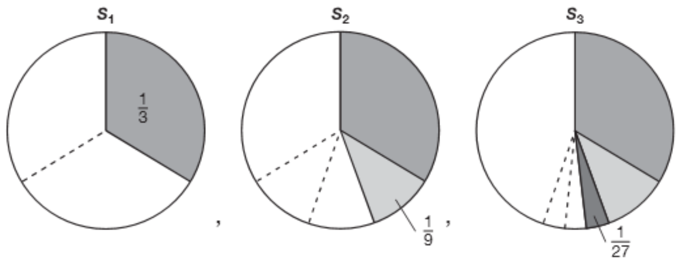
Show that both representations for S_n are equivalent.

PROBLEM 2 To Infinity, and Beyond!

In the previous problem, you saw how an infinite geometric series can have a finite sum. Let's see if this is the case for *any* infinite geometric series.

1. Examine the given formula and accompanying diagram for each infinite geometric series. Identify both r and g_1 for each series. Then, determine if the sum is infinite or finite. If the sum is finite, estimate it.

   a. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i$



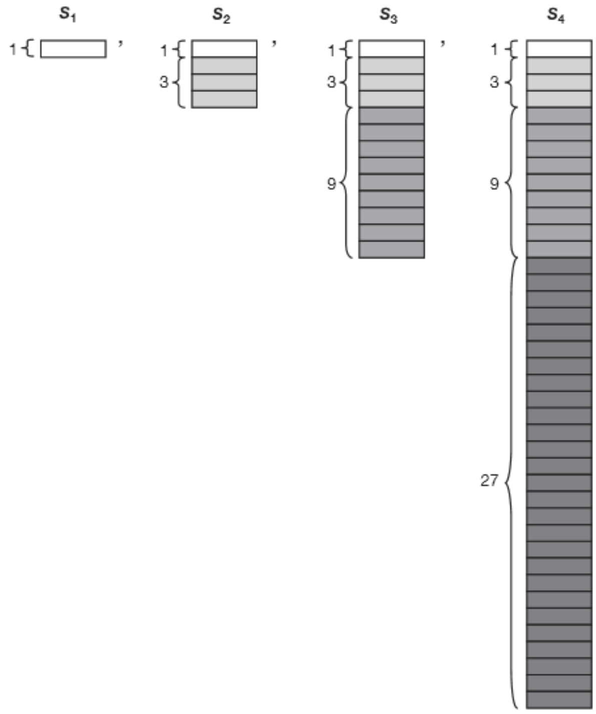
$r =$ ______________________________

$g_1 =$ ______________________________

$S =$ ______________________________

6

b. $1 + 3 + 9 + 27 + \cdots = \sum_{i=1}^{\infty} \frac{1}{3}(3)^i$

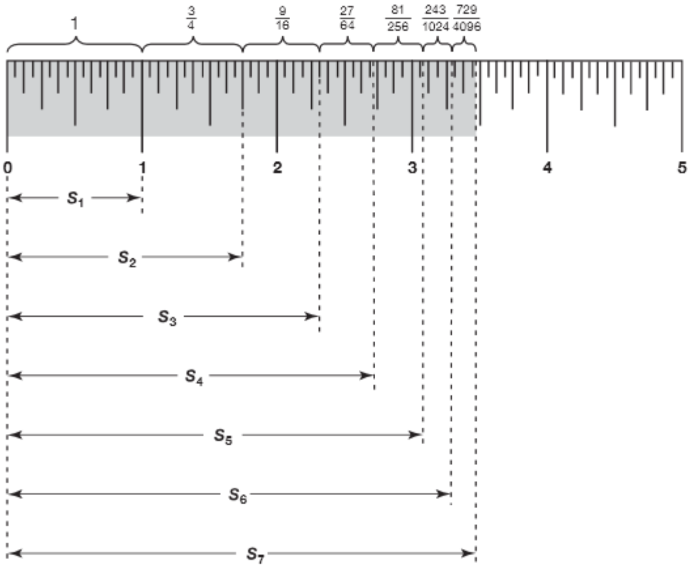


$r =$ ____________________

$g_1 =$ ____________________

$S =$ ____________________

c. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256} + \frac{243}{1024} + \frac{729}{4096} + \cdots = \sum_{i=1}^{\infty} \frac{4}{3}\left(\frac{3}{4}\right)^i$

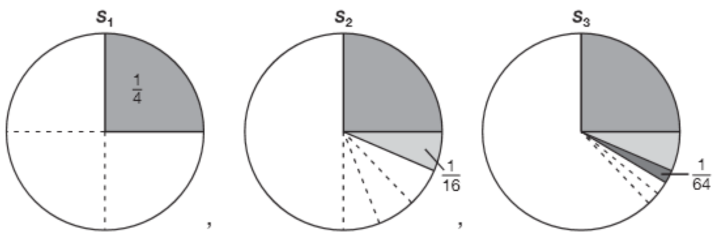


$r =$ ____________________

$g_1 =$ ____________________

$S =$ ____________________

d. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i$



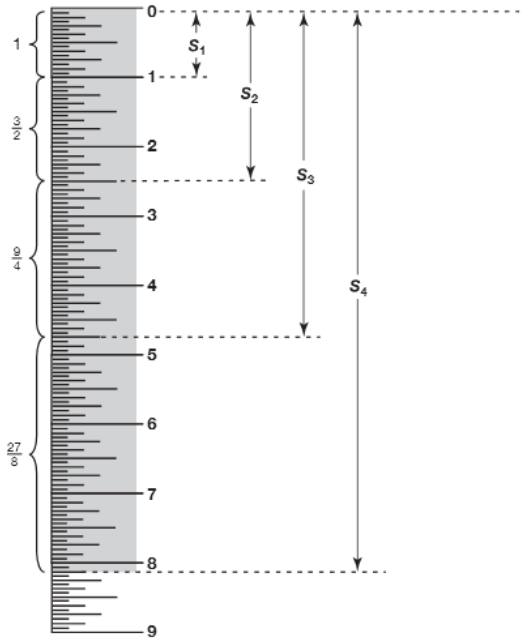
$r =$ ____________________

$g_1 =$ ____________________

$S =$ ____________________

6

e. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{3}{2}\right)^i$



r = ______________________________

g_1 = ______________________________

S = ______________________________

2. Analyze the common ratio for each series in Question 1.

a. What do you notice about the series with infinite sums?

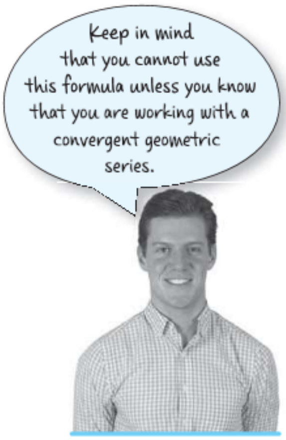
b. What do you notice about the series with finite sums?

A **convergent series** is an infinite series that has a finite sum. A **divergent series** is an infinite series that does not have a finite sum. If a series is divergent, the sum is infinity.

The formula to compute a convergent geometric series S is shown.

$$S = \frac{g_1}{1 - r}$$

Notice that S denotes the sum of an *infinite* series. This notation should not be confused with S_n, which represents the sum of the nth term of a series.



3. Consider each infinite geometric series from Question 1. Determine whether each series is convergent or divergent, and explain how you know. If a series is convergent, use the formula to compute the sum. If a series is divergent, write infinity.

a. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i$

Convergent or divergent? ________________

Explanation:

S = ______________________

b. $1 + 3 + 9 + 27 + \cdots = \sum_{i=1}^{\infty}\frac{1}{3}(3)^i$

Convergent or divergent? ________________

Explanation:

S = ________________

6

c. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256} + \frac{243}{1024} + \frac{729}{4096} + \cdots = \sum_{i=1}^{\infty} \frac{4}{3}\left(\frac{3}{4}\right)^i$

Convergent or divergent? ___________________

Explanation:

$S =$ ___________________________

d. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i$

Convergent or divergent? ___________________

Explanation:

$S =$ ___________________________

e. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{3}{2}\right)^i$

Convergent or divergent? ___________________

Explanation:

$S =$ ___________________

4. Zoe computed the infinite geometric series.

$\frac{5}{8} + \frac{25}{32} + \frac{125}{128} + \frac{625}{512} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2}\left(\frac{5}{4}\right)^i$

Zoe

The formula to compute the series is $S = \frac{g_1}{1 - r}$.

In this series, $g_1 = \frac{5}{8}$ and $r = \frac{5}{4}$.

So, $S = \frac{\frac{5}{8}}{1 - \frac{5}{4}} = \frac{\frac{5}{8}}{-\frac{1}{4}} = \frac{5}{8}\left(-\frac{4}{1}\right) = -\frac{5}{2}$.

Explain why Zoe is incorrect, and then determine the correct sum.

5. Compute each infinite geometric series, if possible.

a. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \frac{9}{100{,}000} + \cdots$

b. $0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

c. 0.9999999 . . .

6

So far in this lesson, you have only seen infinite *geometric* series. What about infinite *arithmetic* series?

6. Consider the statements made by Ronald and Jeremiah about the infinite arithmetic series.

Ronald

Some infinite arithmetic series are convergent, and some are divergent; it all depends on the common difference.

Jeremiah

All infinite arithmetic series are divergent.

Who is correct? Explain your reasoning.

Talk the Talk

Write the formula to compute each type of series.

The First *n*-Terms of an Arithmetic Series:

$S_n =$

The First *n*-Terms of a Geometric Series:

$S_n =$

A Convergent Geometric Series:

$S =$

A Divergent Geometric Series:

$S =$

Be prepared to share your solutions and methods.